X-641-66-65

NASA TM X-55507

THE CONTRIBUTION OF ELECTRON-ELECTRON COLLISIONS TO THE EMISSION OF BREMSSTRAHLUNG BY A PLASMA

BY
THOMAS J. BIRMINGHAM
JOHN M. DAWSON
RUSSELL M. KULSRUD

MARCH 1966

NASA
GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

THE CONTRIBUTION OF ELECTRON-ELECTRON COLLISIONS TO THE EMISSION OF BREMSSTRAHLUNG BY A PLASMA

Thomas J. Birmingham*
National Aeronautics and Space Administration
Laboratory for Theoretical Studies
Goddard Space Flight Center
Greenbelt, Maryland

John M. Dawson and Russell M. Kulsrud
Plasma Physics Laboratory
Princeton University
Princeton, New Jersey

*National Academy of Sciences Research Associate. This paper is in part excerpted from a doctoral thesis presented by T. J. Birmingham to the Physics Department, Princeton University.

ABSTRACT

30330

A new model for treating plasma radiation processes in plasmas is extended to the problem of bremsstrahlung emission from electron-electron collisions. The spectra of longitudinal and transverse waves, including dielectric effects, are derived to lowest non-vanishing (quadrupole) order. A resonance in the spectrum of transverse waves near $\omega = 2\omega_p$, the first harmonic of the electron plasma frequency, is noted. The high frequency form of the transverse spectrum is approximately evaluated and found to be relativistically small by comparison with the dominant electron-ion bremsstrahlung. Finite wave length corrections to the radiation generated by electron-ion collisions are also discussed.

Author

I. INTRODUCTION

In a recent paper[1] (hereafter called Ref. 1) a model describing the emission of radiation by a classical, fully ionized plasma was presented. This model describes, in a simple but self-consistent fashion, plasma modifications of the bare particle bremsstrahlung spectra by considering a) the colliding particles giving rise to the radiation as interacting through a dynamically shielded potential and b) the emission process as taking place in a dispersive plasma medium which alters the propagation characteristics of the waves. Results obtained in the dipole approximation, to which Ref. 1 is restricted, are in agreement with Mercier's[2] and Dupree's[3] calculations, the latter based on a formal hierarchy expansion.

In dipole order the wavelength of the emitted radiation is considered infinitely long compared to a characteristic dimension of its source, which for shielded particle interactions is at most a few electron Debye lengths. In this approximation, only electron-ion collisions produce bremsstrahlung, for in each electron-electron encounter the radiation field of one particle is equal in magnitude and oppositely phased to that of the second.

When the finite ratio of the emission wavelength to the source size is taken into account, collisions between electrons produce a net quadrupole radiation. In addition, there arise finite wavelength corrections to the electron-ion bremsstrahlung spectra.

We address ourselves in this paper to the electron-electron radiation process, applying and extending the techniques developed in Ref. 1. Finite wavelength corrections to the electron-ion spectra have also been derived and a summary of the results of this calculation is included in Section VI.

The collective dielectric aspects of the bremsstrahlung problem are of particular interest in solar physics, where resonances in the spectra of Type II radio bursts at the electron plasma frequency and its harmonic have been explained[4] as, respectively, enhanced electron-ion dipole and electron-electron quadrupole bremsstrahlung from strongly non-thermal plasmas. We shall see presently that a resonance at twice the electron plasma frequency is a natural consequence of considering electron-electron radiative interactions within the framework of our model.

II. THE MODEL

We consider radiation as emanating from test current sources embedded in, interacting with, but logically distinct from an infinite, spatially homogeneous, Vlasov plasma. Initially, the sources are considered as arbitrary current distributions. Later, however, they will be specified as the time varying contributions to the microscopic current arising from collisions between discrete plasma particles.

For the high frequencies of interest ($\omega > \omega_p$, the electron plasma frequency), only the dynamic response of the electron component of the infinite Vlasov plasma need be considered; the ions simply provide a stationary, uniform, neutralizing background. For simplicity we consider the case where no macroscopic fields are present, although the existence of a steady, weak, and uniform magnetic field leads to an interesting modification of the results[5].

Embedded in the plasma, the source current, which we shall describe by a density $\underset{\sim}{j}_s(\underset{\sim}{r},t)$, polarizes the medium and delivers energy to the plasma at a rate

$$P(t) = -\int d^3r\, \underset{\sim}{j}_s(\underset{\sim}{r},t) \cdot \underset{\sim}{E}(\underset{\sim}{r},t), \tag{2.1}$$

where $\underset{\sim}{E}(\underset{\sim}{r},t)$ is the self field arising from $\underset{\sim}{j}_s$, modified by the plasma.

If Eq. (2.1) is averaged over an arbitrarily long (compared with the duration of a typical collision $^1/\omega_p$) interval, $-^T/2$ to $+^T/2$, the average power delivered is just

$$P = -T^{-1}\int d^3r \int_{-T/2}^{+T/2} dt\, \underset{\sim}{j}_s \cdot \underset{\sim}{E}\,. \tag{2.2}$$

When $\mathbf{j}_s$ and $\mathbf{E}$ are expanded as Fourier integrals, Parseval's Theorem can be applied in the limit that T is taken to be infinitely long and Eq. (2.2) cast as an equivalent double integral over the conjugate variables $\mathbf{k}$ and ω,

$$\bar{P} = -(2\pi)^{-4} \lim_{T\to\infty} T^{-1} \int d^3k \int d\omega \; \mathbf{j}_s(\mathbf{k},\omega) \cdot \boldsymbol{\mathcal{E}}(-\mathbf{k},-\omega), \tag{2.3}$$

the Fourier transform of the arbitrary function $g(\mathbf{r},t)$ being defined as

$$g(\mathbf{k},\omega) = \lim_{T\to\infty} \int d^3r \int_{-T/2}^{+T/2} dt \; g(\mathbf{r},t) \exp[-i(\mathbf{k}\cdot\mathbf{r} - \omega t)].$$

The electric field $\boldsymbol{\mathcal{E}}$ is related to $\mathbf{j}_s$ through Maxwell's equations, which can be expressed in the form

$$(k^2 - \omega^2 c^{-2})\boldsymbol{\mathcal{E}} - \mathbf{k}\,\mathbf{k}\cdot\boldsymbol{\mathcal{E}} - 4\pi i \omega c^{-2}(\mathbf{j}_p + \mathbf{j}_s) = 0. \tag{2.4}$$

The quantity $\mathbf{j}_p$ represents the plasma current arising from polarization of the medium by the source. It is the presence of this term which introduces characteristic plasma wave effects (e.g. dielectric shielding of transverse waves and the existence of propagating longitudinal waves) into the problem. Equation (2.4) and an equation of continuity for the total current, plasma plus source, are equivalent to a solution of the full Maxwell equations for $\boldsymbol{\mathcal{E}}$.

For small amplitude waves, $\mathbf{j}_p$ is proportional to $\boldsymbol{\mathcal{E}}$ and for a Vlasov plasma is related to the linear deviation f of the electron distribution from its steady state (in the absence of $\mathbf{j}_s$) value

$$\mathbf{j}_p = -ne \int d^3v \; f\, \mathbf{v}, \tag{2.5}$$

n being the average electron number density and -e the electronic charge. The perturbation f is determined from the linearized Vlasov Equation

$$f = iem^{-1}\left(\underset{\sim}{\mathcal{E}} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c}\right)\cdot\frac{\partial f_0}{\partial \underset{\sim}{v}}\,(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v} + i\varepsilon)^{-1}, \qquad (2.6)$$

m being the electronic mass. The small damping, $i\varepsilon$, has been introduced in the customary way only for mathematical convenience in determining the proper path for contour integration and is ultimately put to zero. If we restrict ourselves to isotropic media so that $f_0(|\underset{\sim}{v}|)$, the magnetic interaction term in Eq. (2.6) vanishes.

Combining Eq. (2.6) with Eq. (2.5) and substituting the result for $\underset{\sim}{\mathcal{J}}_p$ into Eq. (2.4), we solve for $\underset{\sim}{\mathcal{E}}$,

$$\underset{\sim}{\mathcal{E}} = -4\pi i\omega^{-1}k^{-2}\Big\{\left[\underset{\sim}{k}\times(\underset{\sim}{k}\times\underset{\sim}{\mathcal{J}}_s)\right]\left[c^2k^2\omega^{-2} - 1 + \omega_p^2\omega^{-1}\int d^3v\, f_0\,(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v} + i\varepsilon)^{-1}\right]^{-1}$$
$$+\underset{\sim}{k}\,\underset{\sim}{k}\cdot\underset{\sim}{\mathcal{J}}_s\left[1 + \omega_p^2k^{-2}\int d^3v\,\underset{\sim}{k}\cdot\partial f_0/\partial\underset{\sim}{v}\,(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v} + i\varepsilon)^{-1}\right]^{-1}\Big\}, \qquad (2.7)$$

and find that it splits into components longitudinal and transverse to $\underset{\sim}{k}$. In Eq. (2.7), $\omega_p = (4\Pi ne^2m^{-1})^{1/2}$ is the electron plasma frequency.

This value of $\underset{\sim}{\mathcal{E}}$ is in turn introduced into Eq. (2.3) and an expression for $\overline{P}$ quadratic in the source function $\underset{\sim}{\mathcal{J}}_s$ obtained,

$$\overline{P} = -(4\pi^3T)^{-1}i\int d^3k\int d\omega\, k^{-2}\omega^{-1}\left\{\frac{|\underset{\sim}{k}\times\underset{\sim}{\mathcal{J}}_s|^2}{D_T^*(k,\omega)} + \frac{|\underset{\sim}{k}\cdot\underset{\sim}{\mathcal{J}}_s|^2}{D_L^*(k,\omega)}\right\}. \qquad (2.8)$$

It will be henceforth understood that T is an arbitrarily long interval. The average power $\overline{P}$ will be shown to be independent of T (as it should be), so long as T is long. Use has been made in Eq. (2.8) of the fact that since $\underset{\sim}{E}(\underset{\sim}{r},t)$ is a real quantity $\underset{\sim}{\mathcal{E}}(-\underset{\sim}{k},-\omega) = \underset{\sim}{\mathcal{E}}^*(\underset{\sim}{k},\omega)$. The functions D_T and D_L, whose complex

complex conjugate (*) values appear are the usual longitudinal and transverse dielectric functions

$$D_T = 1 - c^2 k^2 \omega^{-2} - \omega_p^2 \omega^{-1} \int d^3v\, f_o\, (\omega - \underset{\sim}{k} \cdot \underset{\sim}{v} + i\varepsilon)^{-1}, \tag{2.9a}$$

$$D_L = 1 + \omega_p^2 k^{-2} \int d^3v\, \underset{\sim}{k} \cdot \partial f_o / \partial \underset{\sim}{v}\, (\omega - \underset{\sim}{k} \cdot \underset{\sim}{v} + i\varepsilon)^{-1}. \tag{2.9b}$$

Because of the assumed isotropy of f_o, they are independent of the direction of $\underset{\sim}{k}$ and each possesses the property $D(k,-\omega) = D^*(k,\omega)$.

It can be readily shown that for a given value of $\underset{\sim}{k}$ the real part of the integrand in Eq. (2.8) is an odd function of ω and vanishes upon integration. The imaginary part of the integrand is, on the other hand, even in ω and non-vanishing so that $\overline{P}$ is (as it must be) a real quantity. We shall consequently multiply the right hand side of Eq. (2.8) by 2 and restrict attention to positive ω-values, interpreting the ω-integrand as the power emission spectrum,

$$\frac{d\overline{P}}{d\omega} = -(2\pi^3 T)^{-1}\, \mathrm{Im} \int d^3k\, k^{-2} \omega^{-1} \left\{ \frac{|\underset{\sim}{k} \times \underset{\sim}{\mathcal{J}}_s|^2}{D_T(k,\omega)} + \frac{|\underset{\sim}{k} \cdot \underset{\sim}{\mathcal{J}}_s|^2}{D_L(k,\omega)} \right\}. \tag{2.10}$$

Equation (2.10) is our fundamental working equation. Resonances in the integrand of Eq. (2.10) arising from the near vanishing of D_L and D_T correspond respectively to the emission of longitudinal and transverse waves. By bremsstrahlung we mean just such wave contributions to the longitudinal and transverse emission spectra when the sources are properly specified.

Longitudinal wave propagation occurs at frequencies close to but slightly higher than ω_p with corresponding wave numbers

$$K_L \approx (\omega^2 - \omega_p^2)^{1/2} (3)^{-1/2} u^{-1}, \tag{2.11}$$

u being the rms thermal velocity. For frequencies much higher than ω_p, the wavelengths of longitudinal oscillations become comparable to or smaller than an electron Debye length ($\lambda_D = u\omega_p^{-1}$), and such waves damp by phase mixing (Landau damping). For a thermal equilibrium plasma Dawson and Oberman[6] point out that only longitudinal oscillations in the approximate frequency interval $\omega_p < \omega \leq 1.4\ \omega_p$ are appreciably undamped.

Transverse waves, on the other hand, propagate at any frequency greater than ω_p with a wave-number frequency relationship

$$K_T \approx (\omega^2 - \omega_p^2)^{1/2} c^{-1}. \tag{2.12}$$

Such transverse waves thus have phase velocities greater than c and hence do not Landau damp.

III. THE SOURCE FUNCTION AND ITS LONG-WAVELENGTH EXPANSION

While the source current $\mathbf{j}_s$ has been arbitrary to this point, we now want to specify it appropriately for the bremsstrahlung process. We first note that bremsstrahlung is emitted by discrete plasma particles and results from interactions among the particles comprising the plasma medium. Because of the mass ratio, the role of ions as the accelerated emitters of such radiation can be neglected and the space Fourier transform of the source current taken as

$$\mathcal{J}_s(\mathbf{k}, t) = -e \sum_e \mathbf{v}_e(t) \exp\left[-i\,\mathbf{k}\cdot\mathbf{r}_e(t)\right], \tag{3.1}$$

the sum being extended over all electrons in the volume of plasma from which the emission is to be computed.

As an aid to isolating the accelerative portion of the current density, we take two time derivatives of Eq. (3.1), Fourier transform in time, and obtain for $\mathcal{J}_s(\mathbf{k},\omega)$ (the quantity which appears in Eq. (2.8)),

$$\mathcal{J}_s(\mathbf{k},\omega) = -\omega^{-2}\ddot{\mathcal{J}}_s(\mathbf{k},\omega) = e\,\omega^{-2}\sum_e \Big[\Big(\ddot{\mathbf{v}}_e - i\,\mathbf{k}\cdot\dot{\mathbf{v}}_e\,\mathbf{v}_e - 2i\,\mathbf{k}\cdot\mathbf{v}_e\,\dot{\mathbf{v}}_e - (\mathbf{k}\cdot\mathbf{v}_e)^2\mathbf{v}_e\Big)\exp(-i\,\mathbf{k}\cdot\mathbf{r}_e)\Big]_\omega . \tag{3.2}$$

An order of magnitude estimate of the terms in this equation indicates that they are in the ratio

$$1 : \frac{ku}{\omega} : \frac{ku}{\omega} : \frac{k^2u^2}{\omega^2} ,$$

where the typical value of $\mathbf{v}_e$ has been taken to be the rms velocity u of the electrons. From the discussion in Section II, we know that the phase velocities of both longitudinal and transverse waves are greater than the electron thermal velocity. Defining, therefore, a smallness parameter $\alpha = ku\omega_p^{-1} = k\lambda_D$, we discard terms of $O\left(\frac{\alpha\omega_p}{\omega}\right)^2$, retaining $O\left(\alpha\,\frac{\omega_p}{\omega}\right)$ terms, since it will be evident shortly that O(1) terms vanish identically for the electron-electron process. [For ω near ω_p, small terms are of $O(\alpha)$; for much higher frequencies they are yet smaller.]

As we are interested in bremsstrahlung from electron-electron collisions, the acceleration $\dot{\mathbf{v}}_e$ and its derivative $\ddot{\mathbf{v}}_e$ can be represented as a sum over the contributions $\dot{\mathbf{v}}_{ee'}$ and $\ddot{\mathbf{v}}_{ee'}$ from individual encounters. To $O\left(\alpha\,\frac{\omega_p}{\omega}\right)$ Eq. (3.2) then becomes

$$\mathcal{J}_s(\mathbf{k},\omega) = e\omega^{-2}\sum_{e\neq e'}\left[\left(\ddot{\mathbf{v}}_{ee'} - i\mathbf{k}\cdot\dot{\mathbf{v}}_{ee'}\mathbf{v}_e - 2i\mathbf{k}\cdot\mathbf{v}_e\,\dot{\mathbf{v}}_{ee'}\right)\exp(-i\mathbf{k}\cdot\mathbf{r}_e)\right]_\omega, \qquad (3.3)$$

and since the sums e and e' extend over the same assemblage of particles, it follows that

$$\begin{aligned}\mathcal{J}_s(\mathbf{k},\omega) = \frac{e}{2\omega^2}\sum_{e\neq e'}\Big[&\left(\ddot{\mathbf{v}}_{ee'} - i\mathbf{k}\cdot\dot{\mathbf{v}}_{ee'}\mathbf{v}_e - 2i\mathbf{k}\cdot\mathbf{v}_e\,\dot{\mathbf{v}}_{ee'}\right)\exp(-i\mathbf{k}\cdot\mathbf{r}_e) \\ &+\left(\ddot{\mathbf{v}}_{e'e} - i\mathbf{k}\cdot\dot{\mathbf{v}}_{e'e}\mathbf{v}_{e'} - 2i\mathbf{k}\cdot\mathbf{v}_{e'}\,\dot{\mathbf{v}}_{e'e}\right)\exp(-i\mathbf{k}\cdot\mathbf{r}_{e'})\Big]_\omega .\end{aligned} \qquad (3.4)$$

Now we expand

$$\exp(-i\mathbf{k}\cdot\mathbf{r}_{e'}) = \sum_{n=0}^{\infty}\frac{\left[i\,\mathbf{k}\cdot(\mathbf{r}_e-\mathbf{r}_{e'})\right]^n}{n!}\exp(-i\mathbf{k}\cdot\mathbf{r}_e), \qquad (3.5)$$

successive terms in the expansion (3.5) are of $O(\alpha)^n$, since we anticipate that shielding (when properly introduced) will limit the range of interaction $\mathbf{r}_e-\mathbf{r}_{e'}$ to a few Debye distances. To order α, $\mathcal{J}_s(\mathbf{k},\omega)$ reduces to

$$\begin{aligned}\mathcal{J}_s(\mathbf{k},\omega) = \frac{e}{2\omega^2}\sum_{e\neq e'}\Big\{\Big[&\ddot{\mathbf{v}}_{ee'} + \ddot{\mathbf{v}}_{e'e}(1+i\mathbf{k}\cdot\mathbf{r}_{ee'}) - i\mathbf{k}\cdot\left(\dot{\mathbf{v}}_{ee'}\mathbf{v}_e + \dot{\mathbf{v}}_{e'e}\mathbf{v}_{e'}\right. \\ &\left.+ 2\mathbf{v}_e\,\dot{\mathbf{v}}_{ee'} + 2\mathbf{v}_{e'}\,\dot{\mathbf{v}}_{e'e}\right)\Big]\exp(-i\mathbf{k}\cdot\mathbf{r}_e)\Big\}_\omega .\end{aligned} \qquad (3.6)$$

where the relative separation vector $\underset{\approx}{r}_{ee'} = \underset{\approx}{r}_e - \underset{\approx}{r}_{e'}$ has been introduced.

For particles e and e' interacting through a Coulomb potential, we have

$$\dot{\underset{\approx}{V}}_{ee'} = \frac{e^2}{m} \frac{\underset{\approx}{r}_{ee'}}{|\underset{\approx}{r}_{ee'}|^3} = -\dot{\underset{\approx}{V}}_{e'e}\,, \tag{3.7}$$

$$\ddot{\underset{\approx}{V}}_{ee'} = \frac{e^2}{m}\left[\frac{(\underset{\approx}{V}_e - \underset{\approx}{V}_{e'})|\underset{\approx}{r}_{ee'}|^2 - 3\underset{\approx}{r}_{ee'}\underset{\approx}{r}_{ee'}\cdot(\underset{\approx}{V}_e - \underset{\approx}{V}_{e'})}{|\underset{\approx}{r}_{ee'}|^5}\right] = -\ddot{\underset{\approx}{V}}_{e'e}\,. \tag{3.8}$$

and the $O(1)$ terms in Eq. (3.6) are identically vanishing. (For electron-ion collisions $\underset{\approx}{\mathcal{J}}_s(\underset{\approx}{k},\omega)$ has a non-zero $O(1)$ component which leads to the dipole bremsstrahlung discussed in Ref. 1.) Substituting from the equations of motion and defining the relative velocity vector $\underset{\approx}{v}_{ee'} = \underset{\approx}{v}_e - \underset{\approx}{v}_{e'}$, we obtain for $\underset{\approx}{\mathcal{J}}_s(\underset{\approx}{k},\omega)$

$$\underset{\approx}{\mathcal{J}}_s(\underset{\approx}{k},\omega) = -\frac{ie^3}{2m\omega^2}\underset{\approx}{k}\cdot\sum_{e\neq e'}\left[\left(2\frac{\underset{\approx}{r}_{ee'}\underset{\approx}{V}_{ee'}}{|\underset{\approx}{r}_{ee'}|^3} + 2\frac{\underset{\approx}{V}_{ee'}\underset{\approx}{r}_{ee'}}{|\underset{\approx}{r}_{ee'}|^3} - 3\frac{\underset{\approx}{r}_{ee'}\underset{\approx}{r}_{ee'}\underset{\approx}{r}_{ee'}\cdot\underset{\approx}{V}_{ee'}}{|\underset{\approx}{r}_{ee'}|^5}\right)\exp(-i\underset{\approx}{k}\cdot\underset{\approx}{r}_e)\right]_\omega$$

$$= \frac{i}{6\omega^2}\underset{\approx}{k}\cdot\sum_{e\neq e'}\left(\dddot{\underset{\approx}{Q}}_{ee'}\right)_\omega\,, \tag{3.9}$$

where, for convenience, we have defined

$$\dddot{\underset{\approx}{Q}}_{ee'} = -\frac{3e^3}{m}\left(2\frac{\underset{\approx}{r}_{ee'}\underset{\approx}{V}_{ee'}}{|\underset{\approx}{r}_{ee'}|^3} + 2\frac{\underset{\approx}{V}_{ee'}\underset{\approx}{r}_{ee'}}{|\underset{\approx}{r}_{ee'}|^3} - 3\frac{\underset{\approx}{r}_{ee'}\underset{\approx}{r}_{ee'}\underset{\approx}{r}_{ee'}\cdot\underset{\approx}{V}_{ee'}}{|\underset{\approx}{r}_{ee'}|^5}\right)\exp(-i\underset{\approx}{k}\cdot\underset{\approx}{r}_e)\,. \tag{3.10}$$

(We shall shortly relate $\dddot{\underset{\approx}{Q}}_{ee'}$ to the third time derivative of the e-e' quadrupole tensor).

Now $\dddot{\underset{\approx}{Q}}_{ee'}$ as it appears in Eq. (3.10) involves the exact orbits of particles e and e'. To obtain the exact orbits, the many body problem must be solved and this is a most difficult task. However, for a range of frequencies, extending from ω_p close up to $\omega_{90^\circ} = u r_{90^\circ}^{-1}$, where r_{90° is the impact parameter at which the average electron undergoes a 90° deflection, the collisions which contribute dominantly to the radiation emission are of the small angle variety. The orbit traversed by each of the colliding particles e and e' in such small angle collisions is, to a good approximation (in the absence of external fields), the superposition of its mean rectilinear motion and a small rapidly fluctuating motion produced by the sum interaction of all other particles of the medium.

This fluctuating motion would itself be a difficult quantity to evaluate. However, we are only interested in the average power radiated. In Eq. (2.10) for the power radiated, $\mathcal{J}_s$ enters quadratically. We must therefore evaluate the average of $\sum \underline{\underline{\dddot{Q}}}_{ee'}$ squared. To do this we make use of the superposition principle for dressed particles as put forth by Hubbard[7] and Rostoker[8]. According to Rostoker we may employ the following mean $\underline{\underline{\dddot{Q}}}_{ee'}$,

$$\underline{\underline{\dddot{Q}}}_{ee'} = \underline{\underline{\dddot{Q}}}_{\bar{e}\bar{e}'} + n\int d^3r_{e''}\int d^3v_{e''}\left[\underline{\underline{\dddot{Q}}}_{e''\bar{e}'}\, f(e''|\bar{e}) + \underline{\underline{\dddot{Q}}}_{\bar{e}e''}\, f(e''|\bar{e}')\right] + n^2\int d^3r_{e''}\int d^3v_{e''}\int d^3r_{e'''}\int d^3v_{e'''}\,\underline{\underline{\dddot{Q}}}_{e''e'''}\, f(e''|\bar{e})\, f(e'''|\bar{e}') \tag{3.11}$$

for computing these averages, treating $\underline{\underline{\dddot{Q}}}_{ee'}$ as uncorrelated with $\underline{\underline{\dddot{Q}}}_{e''e'''}$ (in the squared form of $\mathcal{J}_s$) provided the pair ee' is different from the pair e'' e'''.

A bar (-) in Eq. (3.11) signifies the mean rectilinear motion of the particle(s) is to be used, and thus $\underline{\underline{\dddot{Q}}}_{\bar{e}\bar{e}'}$ is the contribution to $\underline{\underline{\dddot{Q}}}_{ee'}$ due to the direct interaction of e and e' as they follow their rectilinear trajectories. The remainder is the contribution to $\underline{\underline{\dddot{Q}}}_{ee'}$ from shielding, i.e. from all other plasma electrons regarded as field particles interacting with e and e' as the latter move along their rectilinear orbits. A derivation of the superposition principle and a discussion of its application to the bremsstrahlung problem are found in the work of Dawson and Nakayama[9].

The quantity $f(e''|\bar{e})$ is the perturbation in the one particle electron distribution (as a function of the phase space coordinates of e'') arising from the rectilinear motion of e. To obtain f, we solve the test particle problem in which e moves along the trajectory $\underline{r}_e(t) = \underline{r}_{eo} + \underline{v}_e t$ through an infinite, uniform Vlasov plasma consisting of mobile electrons and a smeared out neutralizing ion background. Only the longitudinal interaction between electron e and the plasma is retained, the transverse fields being relativistically small if the thermal energy of the electrons is much less than their rest energy. The value of f so derived is

$$f(e''|\bar{e}) = -\frac{e^2}{2\pi^2 m} \int d^3k' \; \frac{\exp[i\underset{\sim}{k}'\cdot(\underset{\sim}{r}_{e''}-\bar{\underset{\sim}{r}}_e)]}{k'^2 D_L(k', \underset{\sim}{k}'\cdot\bar{\underset{\sim}{v}}_e)} \; \frac{\underset{\sim}{k}'\cdot \partial f_0/\partial \underset{\sim}{v}_{e''}}{(\underset{\sim}{k}'\cdot\bar{\underset{\sim}{v}}_e - \underset{\sim}{k}'\cdot\underset{\sim}{v}_{e''} + i\varepsilon)}, \tag{3.12}$$

$D_L(k', \underset{\sim}{k}' \cdot \bar{\underset{\sim}{v}}_e)$ being the usual longitudinal dielectric function, Eq. (2.9b), evaluated for $\omega = \underset{\sim}{k}'\cdot\bar{\underset{\sim}{v}}_e$.

In the Appendix we have derived $\dddot{\underset{\approx}{Q}}_{ee'}$ in accord with Eq. (3.11) and find

$$\dddot{\underset{\approx}{Q}}_{ee'} = -\frac{3}{2\pi^2}\frac{ie^3}{m} \exp(-i\underset{\sim}{k}\cdot\bar{\underset{\sim}{r}}_e) \int d^3k' \; \frac{\exp(i\underset{\sim}{k}'\cdot\bar{\underset{\sim}{r}}_{ee'})}{k'^2}$$

$$\left[\frac{\underset{\sim}{k}\,\bar{\underset{\sim}{v}}_{e'\perp} + \bar{\underset{\sim}{v}}_{e'\perp}\underset{\sim}{k}'}{D_L(k', -\underset{\sim}{k}'\cdot\bar{\underset{\sim}{v}}_e)} - \frac{\underset{\sim}{k}'\,\bar{\underset{\sim}{v}}_{e\perp} + \bar{\underset{\sim}{v}}_{e\perp}\underset{\sim}{k}'}{D_L(k', \underset{\sim}{k}'\cdot\bar{\underset{\sim}{v}}_{e'})} + \frac{(\underset{\approx}{I}\,k'^2 - 4\underset{\sim}{k}'\underset{\sim}{k}')(\underset{\sim}{k}'\cdot\bar{\underset{\sim}{v}}_{ee'})}{k'^2 D_L(k', -\underset{\sim}{k}'\cdot\bar{\underset{\sim}{v}}_e) D_L(k', \underset{\sim}{k}'\cdot\bar{\underset{\sim}{v}}_{e'})}\right], \tag{3.13}$$

A $\perp$ subscript indicates a vector component perpendicular to $\underset{\sim}{k}'$ and $\underset{\approx}{I}$ is the unit dyadic.

IV. QUADRUPOLE SPECTRA

Only the straight line orbits of the interacting particles e and e' enter Eq. (3.13). When $\mathcal{J}_s(\underset{\sim}{k},\omega)$ appears quadratically, as it does in Eq. (2.10), the interacting particles are now considered uncorrelated and there occurs only a sum over e and e' rather than the ordinary four-fold sum. Further, the term $\exp\left(-i\underset{\sim}{k}\cdot\underset{\sim}{r}_e\right)$ in Eq. (3.13) can be replaced by unity, since in squaring $\mathcal{J}_s$ we obtain a factor $\exp\{i\underset{\sim}{k}\cdot[\underset{\sim}{r}_e(t')-\underset{\sim}{r}_e(t)]\}$. (The two different times, t' and t, appear because the two exponentials occur in different Fourier time transforms), which is approximately unity in the long wavelength approximation.

To shorten the presentation while illustrating the technique, we shall develop only the transverse electron-electron bremsstrahlung spectrum. The longitudinal spectrum is similarly derived, and the result of this calculation will be quoted and discussed.

Substituting the value of $\mathcal{J}_s$ from Eq. (3.9) into Eq. (2.10), we obtain

$$\frac{d\bar{P}^T}{d\omega} = -\frac{1}{72\pi^3 T}\frac{1}{\omega^5}\,\mathrm{Im}\int d^3k\,\frac{1}{k^2 D_T(k,\omega)}\sum_{e\neq e'}\left[\underset{\sim}{k}\times\underset{\sim}{k}\cdot\left(\dddot{\underset{\approx}{Q}}_{ee'}\right)_\omega\right]\cdot\left[\underset{\sim}{k}\times\underset{\sim}{k}\cdot\left(\dddot{\underset{\approx}{Q}}_{ee'}+\dddot{\underset{\approx}{Q}}_{e'e}\right)_{-\omega}\right], \quad (4.1)$$

where $\dddot{\underset{\approx}{Q}}_{e'e} = \dddot{\underset{\approx}{Q}}_{ee'}$. The sums still extend over all electrons except the pairs e = e'. The wave contribution is extracted by integrating locally over the resonance in the $|\underset{\sim}{k}|$ integrand occurring near the $|\underset{\sim}{k}|$ value given by Eq. (2.12) and is

$$\frac{d\bar{P}^T}{d\omega} = \frac{1}{72\pi^2 c^5 T}\,\frac{(\omega^2-\omega_p^2)^{3/2}}{\omega^3}\int d\Omega\sum_{e\neq e'}\left[\hat{\underset{\sim}{k}}\times\hat{\underset{\sim}{k}}\cdot\left(\dddot{\underset{\approx}{Q}}_{ee'}\right)_\omega\right]\cdot\left[\hat{\underset{\sim}{k}}\times\hat{\underset{\sim}{k}}\cdot\left(\dddot{\underset{\approx}{Q}}_{ee'}\right)_{-\omega}\right], \quad (4.2)$$

where $\hat{\underset{\sim}{k}} = \frac{\underset{\sim}{k}}{|\underset{\sim}{k}|}$ is unit vector in the direction of wave propagation. The integrand of Eq. (4.2) displays the angular distribution of the emitted transverse radiation.

Indeed, we now recognize that (as used in Eq. (4.2)) $\dddot{\mathbf{Q}}_{ee'}$ is the third time derivative of the local quadrupole tensor

$$\mathbf{Q}_{ee'} = -\frac{e}{2}\left[3\left(\mathbf{r}_e\mathbf{r}_e + \mathbf{r}_{e'}\mathbf{r}_{e'}\right) - \left(|\mathbf{r}_e|^2 + |\mathbf{r}_{e'}|^2\right)\mathbf{I}\right] \tag{4.3}$$

for Coulomb interacting particles e and e'. [The first two terms reduce to (3.10) upon differentiation, while the terms involving $\mathbf{I}$ do not contribute to (4.2).] The magnetic dipole radiation which ordinarily appears at this level in a multipole expansion[10] vanishes, since the local magnetic dipole moment

$$\mathbf{m}_{ee'} = -\frac{e}{4c}\left[\mathbf{r}_e \times \mathbf{V}_e + \mathbf{r}_{e'} \times \mathbf{V}_{e'}\right] \tag{4.4}$$

is time invariant when particles e and e' interact via a Coulomb force.

The total emission is obtained by integrating Eq. (4.2) over all solid angles

$$\frac{d\bar{P}^T}{d\omega} = \frac{1}{270\pi c^5}\,\frac{1}{T}\,\frac{(\omega^2-\omega_p^2)^{3/2}}{\omega^3}\sum_{e\neq e'}\left\{3\left(\dddot{\mathbf{Q}}_{ee'}\right)_\omega : \left(\dddot{\mathbf{Q}}_{ee'}\right)_{-\omega} - \left(\dddot{q}_{ee'}\right)_\omega\left(\dddot{q}_{ee'}\right)_{-\omega}\right\}, \tag{4.5}$$

where $\dddot{q}_{ee'}$ is a scalar corresponding to Eq. (3.10) formed by replacing the dyadic product by a scalar product. Its screened form is (cf. Eq. 3.13)

$$\dddot{q}_{ee'} = \frac{3}{2\pi^2}\frac{ie^3}{m}\exp\left(-i\mathbf{k}\cdot\bar{\mathbf{r}}_e\right)\int d^3k'\,\frac{\exp\left(i\mathbf{k}'\cdot\mathbf{r}_{ee'}\right)}{k'^2}\,\frac{\mathbf{k}'\cdot\bar{\mathbf{V}}_{ee'}}{D_L(k',-\mathbf{k}'\cdot\bar{\mathbf{V}}_e)D_L(k',\mathbf{k}'\cdot\bar{\mathbf{V}}_{e'})}. \tag{4.6}$$

Fourier transforming Eqs. (3.13) and (4.6) in time, we substitute in Eq. (4.5) and obtain

$$\frac{d\bar{P}^T}{d\omega} = -\frac{e^6}{30\pi^3 m^2 c^5}\,\frac{(\omega^2-\omega_p^2)^{3/2}}{\omega^3}\,\frac{1}{T}\sum_{e\neq e'}\int d^3k'\int d^3k''\,\frac{\exp\left[i\left(\mathbf{k}'+\mathbf{k}''\right)\cdot\left(\mathbf{r}_{e_0}-\mathbf{r}_{e'_0}\right)\right]}{k'^2\,k''^2}$$

$$\left\{3\left[\frac{\mathbf{k}'\bar{\mathbf{V}}_{e'\perp} + \bar{\mathbf{V}}_{e'\perp}\mathbf{k}'}{D_L(\mathbf{k}',-\mathbf{k}'\cdot\bar{\mathbf{V}})} - \frac{\mathbf{k}'\bar{\mathbf{V}}_{e\perp} + \bar{\mathbf{V}}_{e\perp}\mathbf{k}'}{D_L(\mathbf{k}',\mathbf{k}'\cdot\bar{\mathbf{V}}_{e'})} + \frac{(\mathbf{I}\,k'^2 - 4\mathbf{k}'\mathbf{k}')(\mathbf{k}'\cdot\bar{\mathbf{V}}_{ee'})}{k'^2 D_L(\mathbf{k}',-\mathbf{k}'\cdot\bar{\mathbf{V}}_e)D_L(\mathbf{k}',\mathbf{k}'\cdot\bar{\mathbf{V}}_{e'})}\right]:\right.$$

$$\left[\frac{\underset{\sim}{K}''\overline{\underset{\sim}{V}}_{e'\perp} + \overline{\underset{\sim}{V}}_{e'\perp}\underset{\sim}{K}''}{D_L(K'', -\underset{\sim}{K}''\cdot\overline{\underset{\sim}{V}}_e)} - \frac{\underset{\sim}{K}''\overline{\underset{\sim}{V}}_{e\perp} + \overline{\underset{\sim}{V}}_{e\perp}\underset{\sim}{K}''}{D_L(K'', \underset{\sim}{K}''\cdot\overline{\underset{\sim}{V}}_{e'})} + \frac{(\underset{\sim}{I}\,K''^2 - 4\underset{\sim}{K}''\underset{\sim}{K}'')(\underset{\sim}{K}''\cdot\overline{\underset{\sim}{V}}_{ee'})}{K''^2 D_L(K'', -\underset{\sim}{K}''\cdot\overline{\underset{\sim}{V}}_e)D_L(K'', \underset{\sim}{K}''\cdot\overline{\underset{\sim}{V}}_{e'})}\right]$$

$$- \frac{\underset{\sim}{K}'\cdot\overline{\underset{\sim}{V}}_{ee'}\ \ \underset{\sim}{K}''\cdot\overline{\underset{\sim}{V}}_{ee'}}{D_L(K', -\underset{\sim}{K}'\cdot\overline{\underset{\sim}{V}}_e)D_L(K', \underset{\sim}{K}'\cdot\overline{\underset{\sim}{V}}_{e'})D_L(K'', -\underset{\sim}{K}''\cdot\overline{\underset{\sim}{V}}_e)D_L(K'', \underset{\sim}{K}''\cdot\overline{\underset{\sim}{V}}_{e'})}\Bigg\}$$

$$\delta(\omega + \underset{\sim}{K}'\cdot\overline{\underset{\sim}{V}}_{ee'})\,\delta(-\omega + \underset{\sim}{K}''\cdot\overline{\underset{\sim}{V}}_{ee'}). \tag{4.7}$$

To calculate the average power per unit volume emitted over the time interval T, we sum the uncorrelated collisions in the following manner. The number of electrons e' with velocities in the interval $d^3\overline{v}_{e'}$ about $\overline{\underset{\sim}{v}}_{e'}$ which will collide with a given electron e having velocity $\overline{\underset{\sim}{v}}_e$ at an impact distance between $\underset{\sim}{b}_{ee'}$ and $\underset{\sim}{b}_{ee'} + d\underset{\sim}{b}_{ee'}$ in time T is just the number of electrons in the angular element $d\varphi$ of a cylindrical shell of radius $|\underset{\sim}{b}_{ee'}|$ and length $|\underset{\sim}{v}_{ee'}|\,T$,

$$dN = n\left|\overline{\underset{\sim}{V}}_{ee'}\right| T\, d\varphi\, \left|\underset{\sim}{b}_{ee'}\right| d b_{ee'} f_0(\overline{V}_{e'})\, d^3\underset{\sim}{V}_{e'}\,. \tag{4.8}$$

In Eq. (4.8) φ is the angle made by $\underset{\sim}{b}_{ee'}$ with some arbitrary axis in the plane transverse to $\overline{\underset{\sim}{v}}_{ee'}$ ($\underset{\sim}{b}_{ee'}$ and $\overline{\underset{\sim}{v}}_{ee'}$ are orthogonal vectors).

For the total average power per unit volume, we multiply a single term in the sum in Eq. (4.7) by the weighting factor (4.8) and the number density n of electrons e, integrate over all impacts allowable in a small angle theory, and average over the velocity distributions for particles e and e'. Without loss of generality the initial displacement $\underset{\sim}{r}_{eo} - \underset{\sim}{r}_{e'o}$ may be considered as the impact vector for each collision. The transverse spectrum reduces upon performing these operations to

$$\frac{d\bar{P}^T}{d\omega} = \frac{8 m^2 e^6}{15\pi m^2 c^5} \frac{(\omega^2-\omega_p^2)^{3/2}}{\omega^3} \int d^3\bar{v}_e \int d^3\bar{v}_{e'}\, f_0(\bar{v}_e)\, f_0(\bar{v}_{e'}) \int d^3k' \frac{1}{k'^4}$$

$$\left\{ \frac{3k'^2 \bar{v}_{e\perp}^2}{\left|D_L(k', \underset{\sim}{k}' \cdot \underset{\sim}{v}_e)\right|^2} + \frac{8\omega^2}{\left|D_L(k', \underset{\sim}{k}' \cdot \underset{\sim}{v}_e)\right|^2 \left|D_L(k', \underset{\sim}{k}' \cdot \underset{\sim}{v}_{e'})\right|^2} \right\} \delta(\omega + \underset{\sim}{k}' \cdot \underset{\sim}{v}_{e'}). \qquad (4.9)$$

Before discussing Eq. (4.9) we shall write down the parallel expression for the emission of quadrupole longitudinal bremsstrahlung. The longitudinal spectrum is obtained by exactly the same process just discussed for transverse waves: Eq. (3.9) becomes the source current for the longitudinal part of Eq. (2.10), the wave contribution is excerpted by local integration over $|\underset{\sim}{k}|$ values near that given by Eq. (2.11), the angular distribution of radiation is integrated over, and finally collisions are summed. The emergent result,

$$\frac{d\bar{P}^L}{d\omega} = \frac{8 m^2 e^6}{15\pi m^2 (3^{1/2} u)^5} \frac{(\omega^2-\omega_p^2)^{3/2}}{\omega^3} \int d^3\bar{v}_e \int d^3\bar{v}_{e'}\, f_0(\bar{v}_e)\, f_0(\bar{v}_{e'}) \int d^3k' \frac{1}{k'^4}$$

$$\left\{ \frac{2k'^2 \bar{v}_{e\perp}^2}{D_L(k', \underset{\sim}{k}' \cdot \underset{\sim}{v}_e)} + \frac{23\,\omega^2}{4\left|D_L(k', \underset{\sim}{k}' \cdot \underset{\sim}{v}_e)\right|^2 \left|D_L(k', \underset{\sim}{k}' \cdot \underset{\sim}{v}_{e'})\right|^2} \right\} \delta(\omega + \underset{\sim}{k}' \cdot \underset{\sim}{v}_{e'}), \qquad (4.10)$$

is similar in structure to the transverse spectrum, differing by a wavelength ratio $(cu^{-1}\, 3^{-1/2})^5$ [cf. Eqs. (2.11) and (2.12)] and numerical factors. These numerical factors arise for two reasons: first, there are two transverse polarizations as opposed to a single longitudinal polarization, and second, for any given collision the Doppler shift (due to the mean motion of the two electrons through the plasma) for the finite wavelength radiation differs for longitudinal and transverse waves.

V. COMMENTS ON THE QUADRUPOLE EMISSION

The first term in both spectra exhibits the customary logarithmic divergence for large values of $|\underline{k}'|$ (small impact distances) and must be cut off at a value $^1/b_{min}$ consistent with the small angle scattering approximation. Because of the slow logarithmic nature of the divergence, it makes little difference whether b_{min} is taken as the typical 90° deflection impact distance $\left(b_{min} \approx \frac{e^2}{mu^2}\right)$ or the DeBroglie wavelength $\left(b_{min} \approx \frac{\hbar}{mu}\right)$ for an average event. From the nondivergent term, little contribution is obtained for large values of $|\underline{k}'|$, so that the integral may here be extended to ∞ with but negligible error.

The $\underline{k}'$ integrands of Eqs. (4.9) and (4.10) exhibit resonances which correspond to the interaction of electrons through self-generated plasma oscillations. These resonances occur in the small $|\underline{k}'|$ region, $|\underline{k}'|\lambda_D \ll 1$, where Landau damping of the oscillations (i.e. $\mathrm{Im} D_L$) is small. In the region where resonant effects are important, the principal contribution to the $\underline{k}'$ integrals is from the non-divergent terms. The resonance will be most strongly manifest when both $|D_L(k', \underline{k}'\cdot\overline{\underline{v}}_e)|^2$ and $|D_L(k', \underline{k}'\cdot\overline{\underline{v}}_{e'})|^2$ are nearly vanishing or from our discussion in Section II when $\underline{k}'\cdot\overline{\underline{v}}_e \approx \underline{k}'\cdot\overline{\underline{v}}_{e'} \approx \pm\omega_p$. Using the δ-functions, we then conclude that the resonance appears in the emitted spectra at the harmonic $2\omega_p$, because the possibility of emission at the difference frequency $\omega \approx 0$ is excluded by the shielding factor $(\omega^2 - \omega_p^{\,2})^{3/2}$. Since longitudinal waves of frequency $\omega \cong 2\omega_p$ are strongly Landau damped (cf. the discussion in Section II), we expect this non-linear resonance to appear only in the transverse bremsstrahlung spectrum.

Tidman and Dupree[4] have studied the enhancement of the transverse bremsstrahlung spectrum near $\omega = 2\omega_p$ and have found that certain electron distributions composed of a tenuous energetic component coexisting with a thermal background can exhibit a significant resonance. The observed resonance around $\omega = 2\omega_p$ in the spectra of Type II solar radio bursts has thus been explained as arising from enhanced electron-electron bremsstrahlung[11]. The accompanying resonance in Type II spectra at ω_p is explained as similarly enhanced electron-ion bremsstrahlung and occurs dominantly in dipole order. These Tidman-Dupree results are included in the transverse dipole spectrum of Ref. 1 and can be extracted by local integration near $\omega = \omega_p$ of Eq. (4.13) of that paper.

For thermal equilibrium

$$f_0(\bar{V}_e) = (2\pi u^2)^{-3/2} \exp - \bar{V}_e^2/2u^2 \tag{5.1}$$

and the transverse and longitudinal spectra become

$$\frac{d\bar{P}^T}{d\omega} = \frac{32}{15\pi}\frac{m^2e^6}{m^2c^5}\frac{(\omega^2-\omega_p^2)^{3/2}}{\omega^3}u\int dk'\,\frac{1}{k'^3}\int_{-\infty}^{+\infty} dv\,\exp-\left(\frac{1}{4k'^2}+v^2\right)\left(\frac{3k'^2}{|D_L^+|^2}+\frac{4}{|D_L^+|^2|D_L^-|^2}\right) \tag{5.2}$$

and

$$\frac{d\bar{P}^L}{d\omega} = \frac{32}{15\pi}\frac{m^2e^6}{m^2(3^{1/2}u)^5}\frac{(\omega^2-\omega_p^2)^{3/2}}{\omega^3}u\int dk'\,\frac{1}{k'^3}\int_{-\infty}^{+\infty} dv\,\exp-\left(\frac{1}{4k'^2}+v^2\right)\left(\frac{2k'^2}{|D_L^+|^2}+\frac{23}{8|D_L^+|^2|D_L^-|^2}\right). \tag{5.3}$$

where, following the notation of Eq. (2.9b),

$$D_L^{\pm} = D_L\left[k'\frac{\omega}{u},\ \omega\left(\frac{1}{2} \pm k'v\right)\right]. \tag{5.4}$$

The integration variables in Eqs. (5.2) and (5.3) have been non-dimensionalized, so that the logarithmically divergent term must now be cut off at a value $k'_{max} \approx \frac{u}{\omega b_{min}}$.

For steady state thermal equilibrium conditions, the emissions of both transverse and longitudinal electron-electron bremsstrahlung are related to their absorption via collision processes through Kirchhoff's Law. Electron-electron collisional damping may be thought of as a viscous effect, since it is only present when the finite wavelength of the spectral components is taken into consideration so that a macroscopic shearing of the electrons (regarded as a fluid) is effected. Alternatively, it may be looked upon as a finite wave resistance due to electron-electron collision processes.

This latter approach has been taken by DuBois and Gilinsky[12], who have calculated the dissipative conductivity component which results from thermal equilibrium electron-electron collisions. The conductivity components for transverse and longitudinal waves which we calculate from Eqs. (5.2) and (5.3) (The calculation is the same as that carried out in Ref. 1 for steady state electron-ion dipole processes.) are in exact agreement with those of DuBois and Gilinsky (including the logarithmic term).

Figure 1 is a graph of the integral

$$J = \int_0^{\infty} dk' \frac{1}{k'^3} \int_{-\infty}^{+\infty} dv\, \exp -\left(\frac{1}{4k'^2} + v^2\right) \frac{1}{|D_L^+|^2 |D_L^-|^2}$$

common to Eqs. (5.2) and (5.3) and normalized to its $D_L \rightarrow 1$ value of $2(\Pi)^{1/2}$.

Because the logarithmically divergent terms in Eqs. (5.2) and (5.3) are weighted toward large k', where the D_L's are close to unity, this integral displays the primary effect of shielding for thermal equilibrium electron-electron events.

As might be expected, shielding becomes decreasingly important as ω progresses higher above ω_p. This decrease results from the fact that collisions at impact distances comparable with a Debye length (where screening is most effective) take place too slowly to contribute to the radiation at high frequencies. High frequency spectral components are emitted in closer impact collisions which are essentially two body in nature. A small but perceptible inflection is noted in Fig. 1 near $\omega = 2\omega_p$. This is the aforementioned collective resonance, which for thermal equilibrium conditions adds but a minute contribution to the overall emission because of the low level of longitudinal, Cerenkov type oscillations in such plasmas[4]. A curve similar to Fig. (1) has been obtained by DuBois and Gilinsky[12] (cf. Fig. 3 of their paper).

For frequencies $\omega \gtrsim 3\omega_p$, Eq. (4.9) can be approximately evaluated by ignoring shielding effects, i.e. by replacing $(\omega^2 - \omega_p^2)^{3/2}$ by ω^3 and letting $D_L \rightarrow 1$,

$$\frac{d\bar{p}^T}{d\omega} \approx \frac{64 m^2 e^6}{15 m^2 c^5} \int d\bar{v}_e \int dk' \frac{1}{k'^3} F(\bar{v}_e) F'\left(\bar{v}_e + \frac{\omega}{k}\right)\left[k'^2 u^2 + 4\omega^2\right] . \qquad (5.5)$$

In Eq. (5.5) the F's are one dimensional electron distributions, and use has been made of the fact that for isotropic distributions

$$\int d^2 v_{e\perp}\, v_{e\perp}^2 f_o = \frac{2}{3} u^2 F . \qquad (5.6)$$

Further evaluation of (5.5) depends on the explicit form of F. For purposes of a rough estimate, however, we assume that distributions of interest can be approximated by

$$F(\bar{v}_e) = \frac{1}{2u} \qquad |\bar{v}_e| < u,$$

$$= 0 \text{ otherwise} . \tag{5.8}$$

Equation (5.5) then becomes

$$\frac{dP^T}{d\omega} = \frac{16\, m^2 e^6}{15\, m^2 c^5}\, u \left\{ 2 \ln \frac{2u}{\omega b_{min}} + \frac{10}{3} \right\} . \tag{5.7}$$

For frequencies at which the theory is valid, $\omega \lesssim .1\omega_{90^\circ}$, the first term in Eq. (5.7) dominates. (Toward the high frequency end, the $^{10}/_3$ may add a factor of ~ 2 to our estimate.) Comparing this logarithmically dominant portion of (5.7) with the high frequency form of the transverse dipole spectrum as derived from Eq. (4.13) of Ref. 1, we find that the ratio, quadrupole to dipole,

$$\frac{P^{IV}}{P^{II}} \approx \frac{2}{5Z} \frac{u^2}{c^2}$$

(Z is the ionic charge), is small even for rather energetic electron distributions. We conclude, therefore, that except for those electron distributions which lead to a considerable enhancement of the bremsstrahlung in the neighborhood of $\omega = 2\omega_p$ via strong collective effects, radiation loss due to electron-electron collisions is negligibly small in the frequency interval considered.

VI. CONCLUDING REMARKS

Before summarizing, we would discuss briefly two points: k^2 bremsstrahlung corrections from electron-ion collisions and the role of relativistic kinematics in the higher order bremsstrahlung emission.

Electron-Ion k^2 Corrections

A treatment very similar to that presented in the preceding sections has been carried out to assess the effects of electron-ion collisions on higher order longitudinal and transverse bremsstrahlung spectra. In this problem it is important to carry the electron-ion source current to octupole order, for in Eq. (2.10) there exists a finite coupling between the dipole and octupole current densities. This dipole-octupole coupling leads to an emission comparable with that obtained from the squared form of the quadrupole $\underset{\sim}{j}_s$. [For isotropic f_0, the dipole-quadrupole interaction vanishes because of the angular symmetry in the integrand of Eq. (2.10)]. The calculation has been carried out assuming uncorrelated ions and for an electron-ion Coulomb interaction force. The superposition principle has also been used to properly account for orbit fluctuations important in distant encounters.

As in the dipole case, a resonance near $\omega = \omega_p$ is found. This resonance arises from a non-linear coupling between the electron plasma oscillation ($\omega \approx \omega_p$) and the ion wave ($\omega \approx 0$) associated with each of the interacting particles.

In thermal equilibrium, the k^2 electron-ion bremsstrahlung corrections can be compared (via Kirchhoff's Law) with the finite wavelength conductivity corrections calculated by Berk[13] for electron-ion collisions. To logarithmically dominant order our emission results lead in this case to expressions for the dissipative conductivity components which agree with Berk's calculations for both transverse and longitudinal waves.

For high frequencies the k^2 correction is small compared with the dipole emission in the approximate ratio

$$\frac{P^{IV}}{P^{II}} = \frac{4}{5}\frac{u^2}{c^2} \tag{6.1}$$

Relativistic Particle Dynamics

We have seen in Eqs. (5.8) and (6.1) that the transverse spectrum arising from electron-electron collisions and the finite wavelength correction to the electron-ion emission are both relativistically small in comparison with the electron-ion dipole emission. The question now arises as to whether inclusion of relativistic effects in the particle equations of motion (i.e. replacing Eqs. (3.7) and (3.8) and the corresponding electron-ion forms by their relativistic generalizations, including the full electromagnetic interaction between particles) generates corrections in dipole order of the same magnitude as those we have calculated.

We argue that relativistic dipole corrections to the electron-electron transverse spectrum will be of $O\left(\frac{u^4}{c^4}\right)$ and hence generally smaller than those which we have considered. This follows from the fact that the dipole power emission is proportional to $|\ddot{\mathbf{p}}(\omega)|^2$ (see Ref. 1), where $\mathbf{p}$ is the local dipole moment for each interaction. To lowest non-vanishing order $\ddot{\mathbf{p}} \sim \frac{u^2}{c^2}$ and hence the dipole power is of $O\left(\frac{u^4}{c^4}\right)$.

For electron-ion encounters, however, we do expect corrections to the dipole spectrum of $O\left(\frac{u^2}{c^2}\right)$. This follows from the fact that $\ddot{\mathbf{p}}(\omega)$ now has an $O(1)$ contribution, so that $|\ddot{\mathbf{p}}(\omega)|^2$ can have an $O\left(\frac{u^2}{c^2}\right)$ term for this type interaction. Relativistic consideration of the problem will lead to dipole electron-ion corrections comparable with those given by Eq. (6.1). This problem is worthy of pursuit and can perhaps be solved by a modification of the method used in this paper.

SUMMARY

Extending the model developed in Ref. 1 to include sources due to the lowest order form of the accelerative current density arising from electron-electron interactions, we have succeeded in obtaining expressions for the longitudinal and transverse quadrupole bremsstrahlung spectra. Shielding of the electrons and the accelerations of the shielding clouds are included, so that the low frequency forms of the spectra are properly modified. Similar methods can be applied to obtain finite wavelength corrections to the electron-ion emission.

The principal features of the spectra thus obtained are:

1.) The transverse electron-electron spectrum exhibits a resonance near $\omega = 2\omega_p$. The resonant emission is small in thermal equilibrium but can be considerably enhanced for non-equilibrium situations, as pointed out by Tidman and Dupree[4].

2.) The thermal equilibrium forms of the spectra lead upon application of Kirchhoff's Law to conductivity values identical with those derived by DuBois and Gilinsky[12].

3.) The high frequency form of the transverse spectrum is approximately evaluated and found to be relativistically small compared with the dominant electron-ion dipole emission.

4.) Finite wavelength corrections to the bremsstrahlung spectra for Coulomb electron-ion interactions are quoted. These spectra exhibit a resonance near $\omega \approx \omega_p$. For thermal equilibrium plasmas, the power emissions are identically related to Berk's[13] conductivity corrections. At high frequencies, where explicit evaluation is possible, the transverse emission is again relativistically small compared with the dominant dipole power.

Acknowledgements

This work was in the main carried out at the Plasma Physics Laboratory, Princeton University. Helpful discussions with several of the theoretical group there, particularly Drs. J. L. Johnson and C. R. Oberman, are gratefully acknowledged. Mr. Henry Fallon of the Plasma Physics Laboratory Computing Section performed the numerical computation.

One of us (T.J.B.) has also enjoyed several interesting talks with Dr. D. A. Tidman of the Institute of Fluid Dynamics, University of Maryland, concerning the astrophysical implications of the calculations described here.

Support of this work at Princeton was through AEC Contract No. AT(30-1)-1238 and use was made of computer facilities supported in part by National Science Foundation Grant NSF-GP579.

Appendix

In this appendix $\dddot{\mathbf{Q}}_{ee'}$ is calculated according to the prescription of Eq. (3.11). From the form of $\dddot{\mathbf{Q}}_{ee'}$, Eq. (3.10), we note that only the terms $\mathbf{r}_{ee'}\,\mathbf{v}_{ee'}|\mathbf{r}_{ee'}|^{-3}\exp\{-i\mathbf{k}\cdot\mathbf{r}_e\}$ and $\mathbf{r}_{ee'}\,\mathbf{r}_{ee'}(\mathbf{r}_{ee'}\cdot\mathbf{v}_{ee'})|\mathbf{r}_{ee'}|^{-5}\exp\{-i\mathbf{k}\cdot\mathbf{r}_e\}$ need be worked out, since $\mathbf{v}_{ee'}\,\mathbf{r}_{ee'}|\mathbf{r}_{ee'}|^{-3}\exp\{-i\mathbf{k}\cdot\mathbf{r}_e\}$ may then be obtained by transposing the first dyadic.

Using Eq. (3.12) for f, we write

$$
\begin{aligned}
\frac{\mathbf{r}_{ee'}\mathbf{v}_{ee'}}{|\mathbf{r}_{ee'}|^3}\exp(-i\mathbf{k}\cdot\mathbf{r}_e) = {} & \frac{\bar{\mathbf{r}}_{ee'}\bar{\mathbf{v}}_{ee'}}{|\bar{\mathbf{r}}_{ee'}|^3}\exp(-i\mathbf{k}\cdot\bar{\mathbf{r}}_e) - \frac{\omega_p^2}{(2\pi)^3}\int d^3x\int d^3w\int d^3k' \\
& \left\{\frac{(\mathbf{x}-\bar{\mathbf{r}}_{e'})(\mathbf{w}-\bar{\mathbf{v}}_{e'})}{|\mathbf{x}-\bar{\mathbf{r}}_{e'}|^3}\,\frac{\exp\{-i[\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot(\mathbf{x}-\bar{\mathbf{r}}_e)]\}}{k'^2 D_L(k',\mathbf{k}'\cdot\bar{\mathbf{v}}_e)}\,\frac{\mathbf{k}'\cdot\partial f_0/\partial\mathbf{w}}{\mathbf{k}'\cdot(\bar{\mathbf{v}}_e-\mathbf{w})+i\varepsilon}\right. \\
& \left.+\frac{(\bar{\mathbf{r}}_e-\mathbf{x})(\bar{\mathbf{v}}_e-\mathbf{w})}{|\bar{\mathbf{r}}_e-\mathbf{x}|^3}\,\frac{\exp\{-[\mathbf{k}\cdot\bar{\mathbf{r}}_e-\mathbf{k}'\cdot(\mathbf{x}-\bar{\mathbf{r}}_{e'})]\}}{k'^2 D_L(k',\mathbf{k}'\cdot\bar{\mathbf{v}}_{e'})}\,\frac{\mathbf{k}'\cdot\partial f_0/\partial\mathbf{w}}{\mathbf{k}'\cdot(\bar{\mathbf{v}}_{e'}-\mathbf{w})+i\varepsilon}\right\} \\
& +\frac{\omega_p^4}{(2\pi)^6}\int d^3x\int d^3x'\int d^3w\int d^3w'\int d^3k'\int d^3k'' \\
& \left\{\frac{(\mathbf{x}-\mathbf{x}')(\mathbf{w}-\mathbf{w}')}{|\mathbf{x}-\mathbf{x}'|^3}\,\frac{\exp\{-i[\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot(\mathbf{x}-\bar{\mathbf{r}}_e)-\mathbf{k}''\cdot(\mathbf{x}'-\bar{\mathbf{r}}_{e'})]\}}{k'^2k''^2 D_L(k',\mathbf{k}'\cdot\bar{\mathbf{v}}_e)\,D_L(k'',\mathbf{k}''\cdot\bar{\mathbf{v}}_{e'})}\right. \\
& \left.\frac{\mathbf{k}'\cdot\partial f_0/\partial\mathbf{w}}{\mathbf{k}'\cdot(\bar{\mathbf{v}}_e-\mathbf{w})+i\varepsilon}\,\frac{\mathbf{k}''\cdot\partial f_0/\partial\mathbf{w}}{\mathbf{k}''\cdot(\bar{\mathbf{v}}_{e'}-\mathbf{w}')+i\varepsilon}\right\}.
\end{aligned}
\qquad (A.1)
$$

Now in Eq. (A.1) $\exp(-i\mathbf{k}\cdot\mathbf{x})$ may everywhere be replaced by $\exp(-i\mathbf{k}\cdot\bar{\mathbf{r}}_e)$, since the interaction range $\mathbf{x}-\bar{\mathbf{r}}_e$ is cut off by shielding at distances of the order of a few Debye lengths and the corrections $[i\mathbf{k}\cdot(\bar{\mathbf{r}}_e-\mathbf{x})]^n (n!)^{-1}$ are thus small in α. (Recall that $\mathbf{k}$ represents the wavenumber of the <u>emitted</u> radiation.)

The velocity integrals in (A.1) can be carried out by noting that for isotropic f_0

$$\int d^3w\, \underset{\sim}{w}\, \frac{\underset{\sim}{k}'\cdot \partial f_0/\partial \underset{\sim}{w}}{\omega - \underset{\sim}{k}'\cdot \underset{\sim}{w} + i\varepsilon} = \frac{\underset{\sim}{k}'\omega}{\omega_p^2}\left[D_L(k',\omega)-1\right], \tag{A.2}$$

where from Eq. (2.9b)

$$\int d^3w\, \frac{\underset{\sim}{k}'\cdot \partial f_0/\partial \underset{\sim}{w}}{\omega - \underset{\sim}{k}'\cdot \underset{\sim}{w} + i\varepsilon} = \frac{k'^2}{\omega_p^2}\left[D_L(k',\omega)-1\right] . \tag{A.3}$$

Thus

$$\frac{\underset{\sim}{r}_{ee'}\underset{\sim}{v}_{ee'}}{|\underset{\sim}{r}_{ee'}|^3} \exp(-i\underset{\sim}{k}\cdot\underset{\sim}{r}_e) = \exp(-i\underset{\sim}{k}\cdot\bar{\underset{\sim}{r}}_e)\Bigg[\!\!\Bigg[\frac{\bar{\underset{\sim}{r}}_{ee'}\bar{\underset{\sim}{v}}_{ee'}}{|\bar{\underset{\sim}{r}}_{ee'}|^3} - \frac{1}{(2\pi)^3}\int d^3k' \int d^3x$$

$$\Bigg\{ \frac{\underset{\sim}{x}-\bar{\underset{\sim}{r}}_{e'}}{|\underset{\sim}{x}-\bar{\underset{\sim}{r}}_{e'}|^3}\left(\frac{\underset{\sim}{k}'\underset{\sim}{k}'\cdot\bar{\underset{\sim}{v}}_e}{k'^2} - \bar{\underset{\sim}{v}}_{e'}\right)\exp\left[i\underset{\sim}{k}'\cdot(\underset{\sim}{x}-\bar{\underset{\sim}{r}}_e)\right]\left(1-\frac{1}{D_L(k',\underset{\sim}{k}'\cdot\bar{\underset{\sim}{v}}_e)}\right)$$

$$+\frac{\bar{\underset{\sim}{r}}_e-\underset{\sim}{x}}{|\bar{\underset{\sim}{r}}_e-\underset{\sim}{x}|^3}\left(\bar{\underset{\sim}{v}}_e - \frac{\underset{\sim}{k}'\underset{\sim}{k}'\cdot\bar{\underset{\sim}{v}}_{e'}}{k'^2}\right)\exp\left[i\underset{\sim}{k}'\cdot(\underset{\sim}{x}-\bar{\underset{\sim}{r}}_{e'})\right]\left(1-\frac{1}{D_L(k',\underset{\sim}{k}'\cdot\bar{\underset{\sim}{v}}_{e'})}\right)$$

$$+\frac{1}{(2\pi)^6}\int d^3k'\int d^3k''\int d^3x\int d^3x'\, \frac{\underset{\sim}{x}-\underset{\sim}{x}'}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3}\left(\frac{\underset{\sim}{k}'\underset{\sim}{k}'\cdot\bar{\underset{\sim}{v}}_e}{k'^2} - \frac{\underset{\sim}{k}''\underset{\sim}{k}''\cdot\bar{\underset{\sim}{v}}_{e'}}{k''^2}\right)$$

$$\exp\left\{i\left[\underset{\sim}{k}'\cdot(\underset{\sim}{x}-\bar{\underset{\sim}{r}}_e)+\underset{\sim}{k}''\cdot(\underset{\sim}{x}'-\bar{\underset{\sim}{r}}_{e'})\right]\right\}$$

$$\left(1-\frac{1}{D_L(k',\underset{\sim}{k}'\cdot\bar{\underset{\sim}{v}}_e)}\right)\left(1-\frac{1}{D_L(k'',\underset{\sim}{k}''\cdot\bar{\underset{\sim}{v}}_{e'})}\right)\Bigg]\!\!\Bigg]. \tag{A.4}$$

The space integrations are in the form of the Fourier transformed Coulomb electric field

$$\int d^3x\, \frac{\underset{\sim}{x}-\underset{\sim}{a}}{|\underset{\sim}{x}-\underset{\sim}{a}|^3}\exp\left\{i\underset{\sim}{k}'\cdot\underset{\sim}{x}\right\} = \frac{4\pi i\underset{\sim}{k}'}{k'^2}\exp\left\{i\underset{\sim}{k}'\cdot\underset{\sim}{a}\right\}. \tag{A.5}$$

When use of (A.5) is made, Eq. (A.4) is easily reducible to a single integration over interaction wavenumbers $\underset{\sim}{k}'$. Further, significant cancellation is induced by expressing $\bar{\underset{\sim}{r}}_{ee'}\bar{\underset{\sim}{v}}_{ee'}|\bar{\underset{\sim}{r}}_{ee'}|^{-3}$ in integral form,

$$\frac{\bar{\mathbf{r}}_{ee'}\bar{\mathbf{V}}_{ee'}}{|\bar{\mathbf{r}}_{ee'}|^3} = -\frac{i}{2\pi^2}\int d^3k' \frac{\mathbf{k}'\bar{\mathbf{V}}_{ee'}}{k'^2} \exp\left\{i\mathbf{k}'\cdot\bar{\mathbf{r}}_{ee'}\right\}. \tag{A.6}$$

In simplest form, we then finally write

$$\frac{\mathbf{r}_{ee'}\mathbf{V}_{ee'}}{|\mathbf{r}_{ee'}|^3}\exp(-i\mathbf{k}\cdot\mathbf{r}_e) = \frac{i\exp(-i\mathbf{k}\cdot\bar{\mathbf{r}}_e)}{2\pi^2}\int d^3k' \frac{\exp(i\mathbf{k}'\cdot\bar{\mathbf{r}}_{ee'})}{k'^2}$$

$$\left[\frac{\mathbf{k}'\bar{\mathbf{V}}_{e'\perp}}{D_L(k',-\mathbf{k}'\cdot\bar{\mathbf{V}}_e)} - \frac{\mathbf{k}'\bar{\mathbf{V}}_{e\perp}}{D_L(k',\mathbf{k}'\cdot\bar{\mathbf{V}}_{e'})} - \frac{\mathbf{k}'\mathbf{k}'\mathbf{k}'\cdot\bar{\mathbf{V}}_{ee'}}{k'^2 D_L(k',\mathbf{k}'\cdot\bar{\mathbf{V}}_{e'})D_L(k',-\mathbf{k}'\cdot\bar{\mathbf{V}}_e)}\right], \tag{A.7}$$

where the $\perp$ subscript designates a vector component perpendicular to $\mathbf{k}'$.

The format for determining $\mathbf{r}_{ee'}\,\mathbf{r}_{ee'}\,\mathbf{r}_{ee'}\cdot\mathbf{v}_{ee'}|\mathbf{r}_{ee'}|^{-5}\exp\{-i\mathbf{k}\cdot\mathbf{r}_e\}$ is quite similar. We write down an equation corresponding to (A.1), carry out the velocity integrals using (A.2) and (A.3), and note that $\exp(-i\mathbf{k}\cdot\mathbf{x})$ can be set equal to $\exp(-i\mathbf{k}\cdot\bar{\mathbf{r}}_e)$ and extracted from the integration process. Corresponding to (A.4), we then obtain

$$\frac{\mathbf{r}_{ee'}\mathbf{r}_{ee'}\mathbf{r}_{ee'}\cdot\mathbf{V}_{ee'}}{|\mathbf{r}_{ee'}|^5}\exp(-i\mathbf{k}\cdot\mathbf{r}_e) = \exp(-i\mathbf{k}\cdot\bar{\mathbf{r}}_e)\left[\frac{\bar{\mathbf{r}}_{ee'}\bar{\mathbf{r}}_{ee'}\bar{\mathbf{r}}_{ee'}\cdot\bar{\mathbf{V}}_{ee'}}{|\bar{\mathbf{r}}_{ee'}|^5} - \frac{1}{(2\pi)^3}\int d^3k'\int d^3x\right.$$

$$\left\{\frac{(\mathbf{x}-\bar{\mathbf{r}}_{e'})(\mathbf{x}-\bar{\mathbf{r}}_{e'})(\mathbf{x}-\bar{\mathbf{r}}_{e'})}{|\mathbf{x}-\bar{\mathbf{r}}_{e'}|^5}\cdot\left(\frac{\mathbf{k}'\mathbf{k}'\cdot\bar{\mathbf{V}}_e}{k'^2}-\bar{\mathbf{V}}_{e'}\right)\exp\left[i\mathbf{k}'\cdot(\mathbf{x}-\bar{\mathbf{r}}_e)\right]\left(1-\frac{1}{D_L(k',\mathbf{k}'\cdot\bar{\mathbf{V}}_e)}\right)\right.$$

$$\left.+\frac{(\bar{\mathbf{r}}_e-\mathbf{x})(\bar{\mathbf{r}}_e-\mathbf{x})(\bar{\mathbf{r}}_e-\mathbf{x})}{|\bar{\mathbf{r}}_e-\mathbf{x}|^5}\cdot\left(\bar{\mathbf{V}}_e-\frac{\mathbf{k}'\mathbf{k}'\cdot\bar{\mathbf{V}}_{e'}}{k'^2}\right)\exp\left[i\mathbf{k}'\cdot(\mathbf{x}-\bar{\mathbf{r}}_{e'})\right]\left(1-\frac{1}{D_L(k',\mathbf{k}'\cdot\bar{\mathbf{V}}_{e'})}\right)\right\}$$

$$+\frac{1}{(2\pi)^6}\int d^3k'\int d^3k''\int d^3x\int d^3x'\,\frac{(\mathbf{x}-\mathbf{x}')(\mathbf{x}-\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^5}\cdot\left(\frac{\mathbf{k}'\mathbf{k}'\cdot\bar{\mathbf{V}}_e}{k'^2}-\frac{\mathbf{k}''\mathbf{k}''\cdot\bar{\mathbf{V}}_{e'}}{k''^2}\right)$$

$$\left.\exp\left\{i\left[\mathbf{k}'\cdot(\mathbf{x}-\bar{\mathbf{r}}_e)+\mathbf{k}''\cdot(\mathbf{x}'-\bar{\mathbf{r}}_{e'})\right]\right\}\left(1-\frac{1}{D_L(k',\mathbf{k}'\cdot\bar{\mathbf{V}}_e)}\right)\left(1-\frac{1}{D_L(k'',\mathbf{k}''\cdot\bar{\mathbf{V}}_{e'})}\right)\right]. \tag{A.8}$$

The space integrals are now in a form slightly different from (A.5). It can be shown, however, that for arbitrary $\underset{\sim}{a}$ and $\underset{\sim}{b}$

$$\int d^3x \frac{(\underset{\sim}{x}-\underset{\sim}{a})(\underset{\sim}{x}-\underset{\sim}{a})(\underset{\sim}{x}-\underset{\sim}{a})\cdot\underset{\sim}{b}}{|\underset{\sim}{x}-\underset{\sim}{a}|^5} \exp(i\underset{\sim}{k}'\cdot\underset{\sim}{x}) =$$

$$\frac{4\pi i}{3k'^2}\exp(i\underset{\sim}{k}'\cdot\underset{\sim}{a})\left[\left(\underset{\approx}{I}-\frac{2\underset{\sim}{k}'\underset{\sim}{k}'}{k'^2}\right)\underset{\sim}{k}'\cdot\underset{\sim}{b}+\underset{\sim}{k}'\underset{\sim}{b}+\underset{\sim}{b}\underset{\sim}{k}'\right] . \quad \text{(A.9)}$$

Equation A.9 is perhaps best evidenced by changing variables to $\underset{\sim}{y} = \underset{\sim}{x} - \underset{\sim}{a}$, expressing $\underset{\sim}{y}\underset{\sim}{y}|\underset{\sim}{y}|^{-5}$ as $-\frac{1}{3}\,\partial/\partial\underset{\sim}{y}\,\frac{1}{|\underset{\sim}{y}|^3}$, integrating by parts, and finally using Eq. (A.5) and its $\underset{\sim}{k}'$ gradient.

For the purpose of combining terms, it is again convenient to express (in analogy with the inversion of Eq. (A.9))

$$\frac{\overline{\underset{\sim}{r}}_{ee'}\overline{\underset{\sim}{r}}_{ee'}\overline{\underset{\sim}{r}}_{ee'}\cdot\overline{\underset{\sim}{V}}_{ee'}}{|\overline{\underset{\sim}{r}}_{ee'}|^5} = -\frac{i}{6\pi^2}\int d^3k' \frac{\exp(i\underset{\sim}{k}'\cdot\overline{\underset{\sim}{r}}_{ee'})}{k'^2}$$

$$\left[\left(\underset{\approx}{I}-\frac{2\underset{\sim}{k}'\underset{\sim}{k}'}{k'^2}\right)\underset{\sim}{k}'\cdot\overline{\underset{\sim}{V}}_{ee'}+\underset{\sim}{k}'\overline{\underset{\sim}{V}}_{ee'}+\overline{\underset{\sim}{V}}_{ee'}\underset{\sim}{k}'\right] . \quad \text{(A.10)}$$

The result obtained after integrating and algebraically simplifying Eq. (A.8) is

$$\frac{\underset{\sim}{r}_{ee'}\underset{\sim}{r}_{ee'}\underset{\sim}{r}_{ee'}\cdot\underset{\sim}{V}_{ee'}}{|\underset{\sim}{r}_{ee'}|^5}\exp(-i\underset{\sim}{k}\cdot\underset{\sim}{r}_e) = \frac{i\exp(-i\underset{\sim}{k}\cdot\overline{\underset{\sim}{r}}_e)}{6\pi^2}\int d^3k' \frac{\exp(i\underset{\sim}{k}'\cdot\overline{\underset{\sim}{r}}_{ee'})}{k'^2}$$

$$\left[\frac{\underset{\sim}{k}'\overline{\underset{\sim}{V}}_{e'\perp}+\overline{\underset{\sim}{V}}_{e'\perp}\underset{\sim}{k}'}{\mathcal{D}_L(k',-\underset{\sim}{k}'\cdot\overline{\underset{\sim}{V}}_e)}-\frac{\underset{\sim}{k}'\overline{\underset{\sim}{V}}_{e\perp}+\overline{\underset{\sim}{V}}_{e\perp}\underset{\sim}{k}'}{\mathcal{D}_L(k',\underset{\sim}{k}'\cdot\overline{\underset{\sim}{V}}_{e'})}-\frac{\underset{\approx}{I}\;\underset{\sim}{k}'\cdot\overline{\underset{\sim}{V}}_{ee'}}{\mathcal{D}_L(k',-\underset{\sim}{k}'\cdot\overline{\underset{\sim}{V}}_e)\mathcal{D}_L(k',\underset{\sim}{k}'\cdot\overline{\underset{\sim}{V}}_{e'})}\right] . \quad \text{(A.11)}$$

From (A.11), (A.7), and the transpose of the latter dyadic form, it is now possible to write down $\dddot{\underset{\approx}{Q}}_{ee'}$ (cf. Eq. 3.10), properly shielded in accordance with the superposition principle,

$$\dddot{\underline{\underline{Q}}}_{ee'} = -\frac{3}{2\pi^2}\frac{ie^3}{m}\exp(-i\underline{k}\cdot\underline{\bar{r}}_e)\int d^3k' \; \frac{\exp(i\underline{k}'\cdot\underline{\bar{r}}_{ee'})}{k'^2}$$

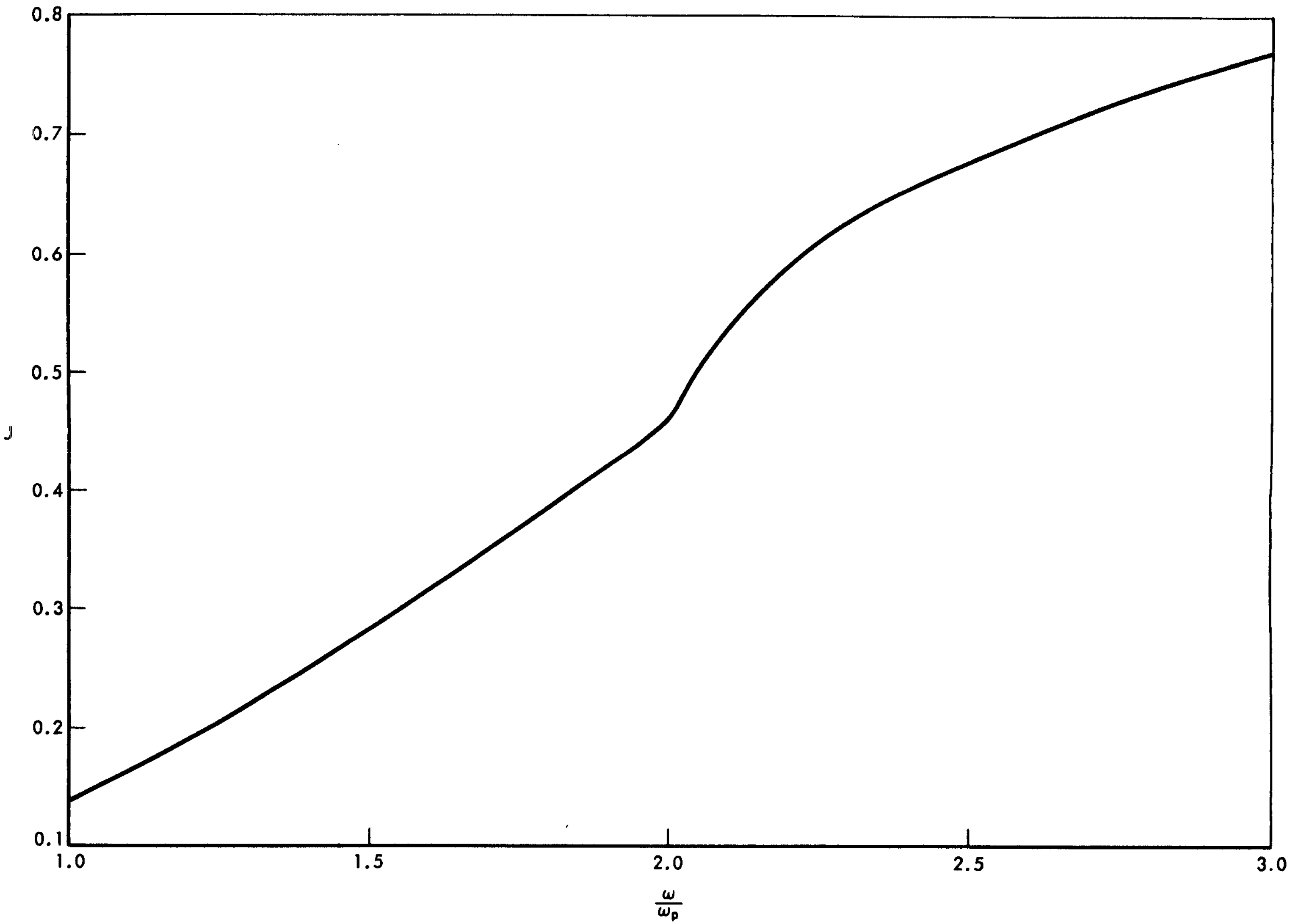
$$\left[\frac{\underline{k}'\underline{\bar{V}}_{e'\perp}+\underline{\bar{V}}_{e'\perp}\underline{k}'}{D_L(k',-\underline{k}'\cdot\underline{\bar{V}}_e)} - \frac{\underline{k}'\underline{\bar{V}}_{e\perp}+\underline{\bar{V}}_{e\perp}\underline{k}'}{D_L(k',\underline{k}'\cdot\underline{\bar{V}}_{e'})} + \frac{(\underline{\underline{I}}\,k'^2-4\underline{k}'\underline{k}')\,\underline{k}'\cdot\underline{\bar{V}}_{ee'}}{k'^2 D_L(k',-\underline{k}'\cdot\underline{\bar{V}}_e)D_L(k',\underline{k}'\cdot\underline{\bar{V}}_{e'})}\right]. \quad (A.12)$$

References

1. T. Birmingham, J. Dawson, and C. Oberman, Phys. Fluids 8, 297 (1965).
2. R. P. Mercier, Proc. Phys. Soc. (London) 83, 819 (1964).
3. T. Dupree, Phys. Fluids 6, 1714 (1963); Phys. Fluids 7, 923 (1964).
4. D. Tidman and T. Dupree, Phys. Fluids 8, 1860 (1965).
5. D. Tidman, T. Birmingham, and H. Stainer, (to be published).
6. J. Dawson and C. Oberman, Phys. Fluids 5, 517 (1962).
7. J. Hubbard, Proc. Roy. Soc. (London) A260, 114 (1961).
8. N. Rostoker, Nuclear Fusion 1, 101 (1960); Phys. Fluids 7, 491 (1964).
9. J. Dawson and T. Nakayama, Phys. Fluids 9, 252 (1966).
10. L. Landau and E. Lifshitz, The Classical Theory of Fields (Addison-Wesley Publishing Company, Inc.; Reading, Massachusetts, 1951), p. 204.
11. The $\omega = 2\omega_p$ peak emission follows from approximately integrating the second term of (4.9) over its resonant values. We are here an inexplicable factor of 2 larger than Tidman-Dupree. In addition we obtain a logarithmically dominant term (in agreement with Ref. 12) absent from the T.-D. formulation. This logarithmic term is perhaps insignificant for calculating the resonant emission.
12. D. F. DuBois and V. Gilinsky, Phys. Rev. 135, A1519 (1964).
13. H. L. Berk, Phys. Fluids 7, 257 (1964).

Figures

1. Numerical evaluation of J as a function of ω in the low frequency region.



Pub. 4843-6